Density Discontinuities at Dent Points Identify the ETI

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Abstract

It is well-known that static labor supply theory predicts that the density of taxable income will feature bunching (a mass point) at a convex kink point in a tax schedule. In this paper, I highlight another prediction of the theory: the density of taxable income should feature a discontinuity at a "dent point" in the tax schedule (a point where the curvature of the tax schedule is discontinuous). As in the case of bunching at kink points, this property of the distribution of taxable income can—in principle—be used to learn about the behavioral response to taxation. In particular, the size of the density discontinuity nonparametrically identifies the local average ETI at the dent point. This result is especially intriguing given that recent work has show that the bunching mass cannot be used to achieve nonparametric identification. However, the applicability of this identification strategy is limited by the relative scarcity of dents points in real world tax or price schedules. However, I show that a variant of the same identification strategy can be used to identify the ETI in a population of "ironing" agents: taxpayers who make decisions as if they believed their marginal tax rate were equal to their average tax rate. I consider possible applications of this version of the strategy to tax analysis and utilities pricing.

1 Unidimensional Heterogeneity

Suppose a population of taxpayers differ only in their ability $w \in \mathbb{R}_{++}$. Each taxpayer chooses a value of taxable income to solve

$$\max_{z} \{z - T(z) - \phi(z; w)\} \tag{1}$$

where $T(\cdot)$ is a nonlinear tax schedule and $\phi(\cdot)$ is disutility of labor supply. Let $\phi(\cdot; w)$ be strictly increasing and strictly convex in z for all w and let $\frac{\partial^2 \phi}{\partial z \partial w} < 0$. Suppose that $T(\cdot)$ is such that for every w there is a unique solution to (1),[1] $z(w)$, which is characterized by the first-order condition

$$1 - T'(z(w)) = \frac{\partial \phi(z(w); w)}{\partial z} \tag{2}$$

and that $z(w)$ is strictly increasing and continuous in w.

Let $w \sim F$. The distribution of taxable income is H must satisfy

$$H(z(w)) = F(w)$$

which implies that the density of taxable income satisfies

$$z'(w)\, h(z(w)) = f(w).$$

Applying the implicit function theorem to the FOC (2), we get

$$z'(w) = -\frac{\frac{\partial^2 \phi(z(w); w)}{\partial z \partial w}}{T''(z(w)) + \frac{\partial^2 \phi(z(w); w)}{\partial z^2}} > 0.$$

Thus

$$h(z(w)) = -\frac{T''(z(w)) + \frac{\partial^2 \phi(z(w); w)}{\partial z^2}}{\frac{\partial^2 \phi(z(w); w)}{\partial z \partial w}} \cdot f(w)$$

Continuous but Kinked MTR

Suppose that at some $z^{\star}$ the marginal tax rate is continuous

$$\lim_{z \to^{+} z^{\star}} T'(z) = \lim_{z \to^{-} z^{\star}} T'(z) \tag{3}$$

[1] This would be true if, for example, $T'(\cdot)$ is weakly increasing.

but the rate of change in the marginal tax rate is discontinuous

$$\lim_{z \to^{+} z^{\star}} T''(z) \neq \lim_{z \to^{-} z^{\star}} T''(z). \tag{4}$$

Let $T''_{\star+} \equiv \lim_{z \to^{+} z^{\star}} T''(z)$ and $T''_{\star-} \equiv \lim_{z \to^{-} z^{\star}} T''(z)$. As well, let $w^{-1}(z)$ be the inverse of $z(w)$ and let $w^{\star} = w^{-1}(z^{\star})$.

Identification Result

The density of taxable income exhibits a discontinuity at $z^{\star}$

$$\lim_{z \to^{+} z^{\star}} h(z) = -\frac{T''_{\star+} + \frac{\partial^2 \phi(z^{\star};w^{\star})}{\partial z^2}}{\frac{\partial^2 \phi(z^{\star};w^{\star})}{\partial z \partial w}} \cdot f(w^{\star}) \neq -\frac{T''_{\star-} + \frac{\partial^2 \phi(z^{\star};w^{\star})}{\partial z^2}}{\frac{\partial^2 \phi(z^{\star};w^{\star})}{\partial z \partial w}} \cdot f(w^{\star}) = \lim_{z \to^{-} z^{\star}} h(z).$$

We can use the left and right limits of the density of taxable income to identify the curvature of the disutility of labor supply for the agents earning $z^{\star}$:

$$\frac{\lim_{z \to^{+} z^{\star}} h(z)}{\lim_{z \to^{-} z^{\star}} h(z)} = \frac{T''_{\star+} + \frac{\partial^2 \phi(z^{\star};w^{\star})}{\partial z^2}}{T''_{\star-} + \frac{\partial^2 \phi(z^{\star};w^{\star})}{\partial z^2}}.$$

That is, if $\lim_{z \to^{+} z^{\star}} h(z)$, $\lim_{z \to^{-} z^{\star}} h(z)$, $T''_{\star+}$, and $T''_{\star-}$ are known, then we can solve the above equation for the unknown parameter $\frac{\partial^2 \phi(z^{\star};w^{\star})}{\partial z^2}$.

From Curvature to ETI

Note, the ETI for taxpayers with ability rate w is

$$\frac{1 - T'(z(w))}{z(w)} \frac{\partial z(w)}{\partial (1 - T')} = \frac{1 - T'(z(w))}{z(w)} \cdot \frac{1}{T''(z(w)) + \frac{\partial^2 \phi(z(w);w)}{\partial z^2}}$$

With a nonlinear tax schedule the realized ETI at a given income depends on the second-derivative of the tax schedule so it to is discontinuous at $z^{\star}$.[2] Nonetheless, since $\frac{\partial^2 \phi(z^{\star};w^{\star})}{\partial z^2}$ is identified, the left and right limits of the ETI are also identified.

Given a specific functional form assumption, we can obtain an estimate of the structural ETI. For instance, suppose that

$$\phi(z;w) \equiv \frac{w}{1 + \frac{1}{e}} \left(\frac{z}{w}\right)^{1+\frac{1}{e}}. \tag{5}$$

[2] Note, this is true even with isoelastic utility but in many presentations of optimal nonlinear income tax results this point is obscured because the type of ETI that appears in the standard ABC formula depends on whether you are integrating over taxable income or skill/ability.

In this case,

$$e=\frac{1-T'\left(z^{\star}\right)}{z^{\star}}\frac{1}{\frac{\partial^{2}\phi\left(z^{\star};w^{\star}\right)}{\partial z^{2}}}.$$

Notice, this identification strategy works if $T''_{\star+}>T''_{\star-}$ or if $T''_{\star+}<T''_{\star-}$. It does however, require that the tax schedule is convex (or, at least, not "too concave").[3]

2 Multidimensional Heterogeneity

It turns out that the identification strategy generalizes to allow for heterogeneous elasticities at $z^{\star}$.

Suppose agents differ in both ability and some other characteristics. Specifically, each agent has some type $(w,\theta)\in\mathbb{R}_{++}\times\Theta$ where Θ may be multidimensional. Let $(w,\theta)\sim F$. The conditional distribution of taxable income is $H\left(\cdot|\theta\right)$ must satisfy

$$H\left(z\left(w,\theta\right)|\theta\right)=F\left(w|\theta\right)$$

Suppose that agent preferences satisfy the single crossing condition within each group θ so that $z_{w}\left(w,\theta\right)>0$ for all (w,θ). Then there exists some inverse function $w^{-1}\left(z,\theta\right)$ such that $z\left(w^{-1}\left(\hat{z},\theta\right),\theta\right)=\hat{z}$ for all $\hat{z}$ and all θ. The conditional CDF of taxable income at z can thus be written as

$$H\left(z|\theta\right)=F\left(w^{-1}\left(z,\theta\right)|\theta\right)$$

and the conditional density is

$$h\left(z|\theta\right)=\frac{\partial w^{-1}\left(z,\theta\right)}{\partial z}f\left(w^{-1}\left(z,\theta\right)|\theta\right).$$

Note, differentiating the condition $z\left(w^{-1}\left(z,\theta\right),\theta\right)=z$ we obtain

$$\frac{\partial w^{-1}\left(z,\theta\right)}{\partial z}=\frac{1}{z_{w}\left(w^{-1}\left(z,\theta\right),\theta\right)}$$

so the conditional density can also be written as

$$h\left(z|\theta\right)=\frac{f\left(w^{-1}\left(z,\theta\right)|\theta\right)}{z_{w}\left(w^{-1}\left(z,\theta\right),\theta\right)}.$$

[3] This is required to ensure taxpayers all have a unique optimum and $z\left(w\right)$ is a strictly increasing, continuous function

Using the implicit function theorem we can provide even more detail

$$z_w\left(w^{-1}(z,\theta),\theta\right) = -\frac{\frac{\partial^2\phi\left(z;w^{-1}(z,\theta),\theta\right)}{\partial z\partial w}}{T''(z) + \frac{\partial^2\phi\left(z;w^{-1}(z,\theta),\theta\right)}{\partial z^2}}.$$

This expression makes it clear that the conditional density of taxable income at z depends on the second derivative of the tax liability function at this point $T''(z)$. Let's define a function which tells us the hypothetical conditional density of taxable income at z when the second derivative is T'', holding all marginal tax rates constant:

$$h(z|\theta;T'') \equiv -\frac{T'' + \frac{\partial^2\phi\left(z;w^{-1}(z,\theta),\theta\right)}{\partial z^2}}{\frac{\partial^2\phi\left(z;w^{-1}(z,\theta),\theta\right)}{\partial z\partial w}} f\left(w^{-1}(z,\theta)|\theta\right).$$

Similarly, the hypothetical unconditional density when the second derivative is T'' can be written as

$$h(z;T'') = -\int \frac{T'' + \frac{\partial^2\phi\left(z;w^{-1}(z,\theta),\theta\right)}{\partial z^2}}{\frac{\partial^2\phi\left(z;w^{-1}(z,\theta),\theta\right)}{\partial z\partial w}} f\left(w^{-1}(z,\theta)|\theta\right) f_\theta(\theta)\,\mathrm{d}\theta$$

Suppose that at some $z^\star$ the tax schedule is continuous differentiable, but has a discontinuous second derivative. In particular, let

$$T''_{\star 0} \equiv \lim_{z\to^- z^\star} T''(z),$$

and

$$T''_{\star 1} \equiv \lim_{z\to^+ z^\star} T''(z).$$

Then the left and right limits of the density of taxable income at $z^\star$ can be written as

$$\lim_{z\to^- z^\star} h(z) = h(z^\star;T''_0),$$

and

$$\lim_{z\to^+ z^\star} h(z) = h(z^\star;T''_1).$$

Note, these limits can be identified from the observed distribution of taxable income. Thus, we can identify their ratio:

$$
\begin{aligned}
\frac{h\left(z^{\star};T_{1}''\right)}{h\left(z^{\star};T_{0}''\right)} &= \frac{T_{1}''\cdot\int\left[\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z\partial w}\right]^{-1}f_{w}\left(w^{-1}\left(z^{\star},\theta\right)|\theta\right)f_{\theta}\left(\theta\right)\mathrm{d}\theta+\int\frac{\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z^{2}}}{\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z\partial w}}f_{w}\left(w^{-1}\left(z^{\star},\theta\right)|\theta\right)f_{\theta}\left(\theta\right)\mathrm{d}\theta}{T_{0}''\cdot\int\left[\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z\partial w}\right]^{-1}f_{w}\left(w^{-1}\left(z^{\star},\theta\right)|\theta\right)f_{\theta}\left(\theta\right)\mathrm{d}\theta+\int\frac{\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z^{2}}}{\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z\partial w}}f_{w}\left(w^{-1}\left(z^{\star},\theta\right)|\theta\right)f_{\theta}\left(\theta\right)\mathrm{d}\theta} \\
&= \frac{1+T_{1}''\cdot\left(\frac{\int\left[\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z\partial w}\right]^{-1}f_{w}\left(w^{-1}\left(z^{\star},\theta\right)|\theta\right)f_{\theta}\left(\theta\right)\mathrm{d}\theta}{\int\left(\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z^{2}}/\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z\partial w}\right)f_{w}\left(w^{-1}\left(z^{\star},\theta\right)|\theta\right)f_{\theta}\left(\theta\right)\mathrm{d}\theta}\right)}{1+T_{0}''\cdot\left(\frac{\int\left[\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z\partial w}\right]^{-1}f_{w}\left(w^{-1}\left(z^{\star},\theta\right)|\theta\right)f_{\theta}\left(\theta\right)\mathrm{d}\theta}{\int\left(\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z^{2}}/\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z\partial w}\right)f_{w}\left(w^{-1}\left(z^{\star},\theta\right)|\theta\right)f_{\theta}\left(\theta\right)\mathrm{d}\theta}\right)}
\end{aligned}
$$

Since T_{1}'' and T_{0}'' are known, we can identify the unknown expression

$$
\begin{aligned}
\frac{\int\left[\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z\partial w}\right]^{-1}f_{w}\left(w^{-1}\left(z^{\star},\theta\right)|\theta\right)f_{\theta}\left(\theta\right)\mathrm{d}\theta}{\int\left(\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z^{2}}/\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z\partial w}\right)f_{w}\left(w^{-1}\left(z^{\star},\theta\right)|\theta\right)f_{\theta}\left(\theta\right)\mathrm{d}\theta} &= \frac{\int\left[\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z^{2}}\right]^{-1}\left(\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z^{2}}/\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(\cdots\right.\right.}{\partial z\partial w}\cdots\right.}{\int\left(\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z^{2}}/\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z\partial w}\right)f_{w}\cdots} \\
&= \frac{\int\left[\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z^{2}}\right]^{-1}h\left(z^{\star}|\theta;0\right)f_{\theta}\left(\theta\right)\mathrm{d}\theta}{\int h\left(z^{\star}|\theta;0\right)f_{\theta}\left(\theta\right)\mathrm{d}\theta} \\
&= \mathbb{E}\left[\left(\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z^{2}}\right)^{-1}|z\left(w,\theta\right)=z^{\star},T''=0\cdots\right.
\end{aligned}
$$

The first line follows from multiplying and dividing the integrand of the numerator by $\frac{\partial^{2}\phi\left(z^{\star};w^{-1}\left(z^{\star},\theta\right),\theta\right)}{\partial z^{2}}$, and the second line follows from the definition of $h\left(z|\theta;T''\right)$.

Further let

$$
\varepsilon^{c}\left(z,\theta;T''\right)\equiv\frac{1-T'\left(z\right)}{z}\cdot\frac{1}{T''+\frac{\partial^{2}\phi\left(z;w^{-1}\left(z,\theta\right),\theta\right)}{\partial z^{2}}}
$$

be the (hypothetical) compensated elasticity of taxable income of the type θ agent at z if the second derivative of the tax schedule at z were T''. The actual or *total response* elasticity, taking into account the curvature of the tax schedule is $\varepsilon^{c}\left(z,\theta;T''\left(z\right)\right)$. The so-called *virtual* or *direct* compensated elasticity is $\varepsilon^{c}\left(z,\theta;0\right)$. The local average virtual elasticity at $z^{\star}$ is what we can identify using the ratio of the densities on either side of $z^{\star}$:

$$
\frac{h\left(z^{\star};T_{1}''\right)}{h\left(z^{\star};T_{0}''\right)}=\frac{1+\frac{z^{\star}T_{1}''}{1-T'\left(z^{\star}\right)}\mathbb{E}\left[\varepsilon^{c}|z=z^{\star},T''=0\right]}{1+\frac{z^{\star}T_{0}''}{1-T'\left(z^{\star}\right)}\mathbb{E}\left[\varepsilon^{c}|z=z^{\star},T''=0\right]}.
$$

This is parameter is still directly policy relevant when the tax schedule is linear. So in the German setting for example, if we applied this identification strategy at the top MTR kink, we could say something about whether the top MTR is revenue-maximizing, since the tax schedule becomes linear after that kink. More generally, it is not possible to transform average virtual elasticities into average total response elasticities. However, note that $\varepsilon^c(z, \theta; T'')$ is strictly decreasing in T''. Thus, whenever $T'' > 0$, we will identify an upper bound on the average total response elasticity:

$$\mathbb{E}\left[\varepsilon^c(z, \theta; T'') \,|z = z^\star\right] < \mathbb{E}\left[\varepsilon^c(z, \theta; 0) \,|z = z^\star\right].$$

If $T'' < 0$ we would instead be obtaining a lower bound on the total response elasticity.

3 Potential Application: What Does Ironing Look Like?

Static, frictionless models of consumer demand or labor supply predict that agents facing a nonlinear price schedule will "bunch" at—or cluster around—convex kinks in their budget constraint. **?** documented evidence of such bunching among self-employed taxpayers at the first kink in the Earned Income Tax Credit schedule in the United States. However, Saez also found very limited evidence of bunching at other kink points or among wage earners.[4] **?** find little evidence of bunching at the vast majority of kink points in the United States income tax schedule. Beyond the US context, among income taxpayers is inconsistently documented and often present only in specific sub-populations of taxpayers.

One possible explanation for the absence of bunching is the so called "ironing" heuristic introduced by **?**. A taxpayer who employs this heuristic acts as if they misperceive their marginal tax rate. In particular, they act as if their perceived tax rate were their average tax rate. **?** present evidence suggesting that as many as 43% of US income taxpayers employ the ironing heuristic. In a parallel literature, several studies of residential electricity and water demand have found little if any evidence of bunching at convex kinks in nonlinear utility price schedules, and have documented behavior consistent with widespread use of the ironing heuristic among utilities consumers [**????**].

A simple reinterpretation of the model discussed in the sections above allows us to apply the identification results presented to the case of ironing agents. Suppose each taxpayer's *experienced utility* is

$$u(z; w) \equiv z - T(z) - \phi(z; w) \tag{6}$$

where $T(\cdot)$ is a nonlinear tax schedule and $\phi(\cdot; w)$ is disutility of labor supply. Let $\phi(\cdot; w)$ be strictly increasing and strictly convex in z for all w and let $\frac{\partial^2 \phi}{\partial z \partial w} < 0$.

[4] Or at least, found that any bunching was small.

Each taxpayer's *decision utility* is

$$u^s(z; w, \bar{z}) \equiv z - A(\bar{z})\, z - \phi(z; w)$$

where $A(\bar{z}) \equiv \frac{T(\bar{z})}{\bar{z}}$ is the average tax liability at a given reference point $\bar{z}$.

Suppose that for each type w,

$$\hat{z}(w, \bar{z}) \equiv \arg\max_z \{z - A(\bar{z})\, z - \phi(z; w)\}.$$

Suppose that $T(\cdot)$ is such that for every w there is a unique solution $z(w)$, which is characterized by

$$z(w) = \hat{z}(w, z(w))$$

or, alternatively, by the first-order condition

$$1 - A(z(w)) = \frac{\partial \phi(z(w); w)}{\partial z}. \tag{7}$$

In this model, the behavior of an ironing agent with respect to a kink point is the same as a standard agent with respect to a dent point. Thus, in principle, for a population composed completely ironing agents, we should expect to see a density discontinuity appear at a tax bracket threshold (a kink point) and the size of this discontinuity could be used to identify a local average ETI for the ironing agents.

This result generates helpful qualitative predictions about the appearance of the distribution of taxable income when ironers are present. Such predictions can be used to help assess the prevalence of ironing behavior in a given population. I use simulations to document what the distribution of taxable income can look like with ironing agents in the presence of optimization error. Interestingly, with moderate optimization error, ironing can give rise to what I call *psuedo-bunching*: a lump in the density of taxable income which an observed might confuse for evidence of a bunching mass which has been diffused by optimization error. Ironing-induced *psuedo-bunching* is right-biased: that is, the apparent bunching occurs predominantly above a convex kink point. This finding may prove helpful to researchers attempting to determine whether their data are consistent with the presence of a large number of ironing agents.

Figure 1 shows the simulated density of taxable income generated by ironing agents in the vicinity of several convex kinks. Figure 2 shows the simulated density of taxable income generated by ironing agents in the vicinity of several non-convex kinks. The first column in both figures shows a simulation with no optimization errors. As you look across each row, more optimization errors is added in each figure as you go from left to right. Note that in some cases, we see something that looks like bunching: *psuedo-bunching.*

(e) $z^\star = 0.6$, $\sigma = 0$

(f) $z^\star = 0.6$, $\sigma = 0.02$

(g) $z^\star = 0.6$, $\sigma = 0.04$

(h) $z^\star = 0.6$, $\sigma = 0.06$

(i) $z^\star = 0.8$, $\sigma = 0$

(j) $z^\star = 0.8$, $\sigma = 0.02$

(k) $z^\star = 0.8$, $\sigma = 0.04$

(l) $z^\star = 0.8$, $\sigma = 0.06$

(m) $z^\star = 1.0$, $\sigma = 0$

(n) $z^\star = 1.0$, $\sigma = 0.02$

(o) $z^\star = 1.0$, $\sigma = 0.04$

(p) $z^\star = 1.0$, $\sigma = 0.06$

(e) $z^\star = 0.6$, $\sigma = 0$

(f) $z^\star = 0.6$, $\sigma = 0.02$

(g) $z^\star = 0.6$, $\sigma = 0.04$

(h) $z^\star = 0.6$, $\sigma = 0.06$

(i) $z^\star = 0.8$, $\sigma = 0$

(j) $z^\star = 0.8$, $\sigma = 0.02$

(k) $z^\star = 0.8$, $\sigma = 0.04$

(l) $z^\star = 0.8$, $\sigma = 0.06$

(m) $z^\star = 1.0$, $\sigma = 0$

(n) $z^\star = 1.0$, $\sigma = 0.02$

(o) $z^\star = 1.0$, $\sigma = 0.04$

(p) $z^\star = 1.0$, $\sigma = 0.06$

References